# SPECTRAL TRIPLES FOR FINITELY PRESENTED GROUPS, INDEX 1

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Abstract. Using a Cayley complex (generalizing the Cayley graph) and Clifford algebras, we are able to give, for a large class of finitely presented groups, a uniform construction of spectral triples with $D_+$ of index 1.

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**Warning 0.1.** *This paper is just a first draft, it contains very few proofs. It is possible that some propositions are false, or that some proofs are incomplete or trivially false.*

## 1. Introduction

In this paper, we define even $\theta$-summable spectral triples for a large class of finitely presented groups such that $D_+$ is index 1. We just generalize the unbounded version of the construction of the Fredholm module for the free group given by Connes [1] and M. Pimsner-Voiculescu [5]. For so, we use the Clifford algebra in the same spirit that Julg-Valette do in [4]. We also use topics in geometric group theory as a Cayley complex (generalizing the Cayley graph).

2000 *Mathematics Subject Classification.* Primary 46L87. Secondary 20F65.

*Key words and phrases.* non-commutative geometry; spectral triple; geometric group theory; Clifford algebra; Cayley graph; Dirac operator; finitely presented groups.

## 2. Basic definitions

**Definition 2.1.** *A spectral triple $(\mathcal{A}, H, D)$ is given by a unital $\star$-algebra $\mathcal{A}$ representated on the Hilbert space $H$, and an unbounded operator $D$, called the Dirac operator, such that:*

(1) *$D$ is self-adjoint.*
(2) *$(D^2+I)^{-1}$ is compact.*
(3) *$\{a \in \mathcal{A} \mid [D,a] \in B(H)\}$ is dense in $\mathcal{A}$.*

*See the article* [6] *of G. Skandalis, dedicated to A. Connes and spectral triple.*

**Definition 2.2.** *A group $\Gamma$ is finitely presented if it exists a finite generating set $S$ and a finite set of relations $R$ such that $\Gamma = \langle S \mid R \rangle$. We always take $S$ equals to $S^{-1}$ and the identity element $e \notin S$ (see* [3] *for more details).*

## 3. Geometric construction

**Definition 3.1.** *Let $\Gamma_n$ be the set of irreducible $n$-blocks, defined by induction:*

- $\Gamma_0 = \Gamma$.
- $\Gamma_1 := \{\{g, gs\} \mid g \in \Gamma, s \in S\}$

*An $(n+2)$-block is a finite set $a$ of $(n+1)$-blocks such that:*

$$\forall b \in a, \forall c \in b, \exists! b' \in a \text{ such that } b \cap b' = \{c\}.$$

*Let $a, a'$ be $n$-blocks then the commutative and associative composition:*

$$a.a' := a \triangle a' = (a \cup a') \backslash a \cap a'$$

*gives also an $n$-block if it's non empty (we take $n \neq 0$).*
*Let $n > 1$, an $n$-block $a''$ is called* ***irreducible*** *if $\forall a, a'$ $n$-blocks:*

$$\text{(1) } a'' = a.a' \Rightarrow card(a) \text{ or } card(a') \geq card(a'')$$

$$\text{(2) } \forall b \in a'',\ b \text{ is an irreducible } (n-1)\text{-block.}$$

- *$\Gamma_{n+2}$ is the set of irreducible $(n+2)$-blocks.*

*Note that if $b \in \Gamma_n$, we call $n$ the* ***dimension*** *of $b$.*

**Definition 3.2.** *An $n$-block is called* ***admissible*** *if it decomposes into irreducibles.*

**Example 3.3.** *Let $\mathbb{Z} = \langle s^{\pm 1} \mid \rangle$ then $a = \{e, s^{10}\}$ is an admissible $1$-block because $a = \{e,s\}.\{s,s^2\}...\{s^9,s^{10}\}$; but, $b = \{\{e,s\},\{e,s^{-1}\},\{s^{-1},s\}\}$ is a non-admissible $2$-block, because there is no irreducible $2$-block in this case.*

**Remark 3.4.** *The graph with vertices $\Gamma_0$ and edges $\Gamma_1$ is the Cayley graph $\mathcal{G}$.*

**Remark 3.5.** *Let $a$ be an $n$-block then $a.a = \emptyset$ and if $a = \{b_1, .., b_r\}$ then $b_i = b_1.b_2...b_{i-1}.b_{i+1}...b_r$ and $b_1.b_2...b_r = \emptyset$.*

**Remark 3.6.** *$\Gamma_{n+1} \neq \emptyset$ iff $\exists r > 1; a_1, ..., a_r \in \Gamma_n$ all distincts with $a_1...a_r = \emptyset$.*

**Remark 3.7.** *Let $\Gamma = \langle S \mid R \rangle$ be a finitely presented group, then $\exists N$ such that $\Gamma_N \neq \emptyset$ and $\forall n > N$, $\Gamma_n = \emptyset$. In fact $N \leq card(S)$*

**Examples 3.8.** *For $\mathbb{F}_r = \langle s_1^{\pm 1}, ..., s_r^{\pm 1} \mid \rangle$, we have $N = 1$.
For $\mathbb{Z}^r = \langle s_1^{\pm 1}, ..., s_r^{\pm 1} \mid s_i s_j s_i^{-1} s_j^{-1}, i, j = 1, ..., r \rangle$, we have $N = r$.
Here an $n$-block ($n \leq r$) is just an $n$-dimensional hypercube.*

**Definition 3.9.** *We define the action of $\Gamma$ on $\Gamma_n$ recursively:*

- *$\Gamma$ acts on $\Gamma_0 = \Gamma$ as: $u_g : h \to g.h$ with $g, h \in \Gamma$.*
- *Action on $\Gamma_{n+1}$: $u_g : a \to g.a = \{g.b \mid b \in a\}$ with $g \in \Gamma$, $a \in \Gamma_{n+1}$.*

*Note that the action is well-defined: $g.\Gamma_n = \Gamma_n$, $\forall g \in \Gamma$.*

**Definition 3.10.** *Let $a$ and $b$ be blocks, then we say that $b \Subset a$ if $b = a$ or if $b \in a$ or if $\exists c \in a$ such that $b \Subset c$ (recursive definition).*

**Definition 3.11.** *Let $n > 1$ then an $n$-block $c$ is **connected** if $\forall b \subset c$: '$b$ is an $n$-block' $\Rightarrow b = c$.*

**Definition 3.12.** *An $n$-block $b$ is called **maximal** if there is no $(n+1)$-block $c$ with $b \in c$. We note $\mathbf{\Gamma_{max}}$ the set of maximal irreducible blocks.*

**Example 3.13.** *Let $\Gamma = \mathbb{Z}^2 \star \mathbb{Z} = \langle s_1^{\pm 1}, s_2^{\pm 1}, s_3^{\pm 1} \mid s_1 s_2 s_1^{-1} s_2^{-1} \rangle$, then $\{e, s_3\}$ is a maximal 1-block, $\{\{e, s_1\}, \{s_1, s_1 s_2\}, \{s_1 s_2, s_2\}, \{s_2, e\}\}$ is a maximal 2-block.*

**Definition 3.14.** *We define the **block lenght** $\ell(.)$ as follows: let $b$ be a block, then $\ell(b)$ is the minimal number of irreducible blocks decomposing a connected admissible block $c$ with $e \Subset c$ and, $b \Subset c$ or $b \cap c \neq \emptyset$.*

**Definition 3.15.** *Let $b$ be a block, then a sequence $(c_1, ..., c_{\ell(b)})$ with $b \Subset c_1$, $e \Subset c_{\ell(b)}$, $c_i$ irreducible and $c_i \cap c_{i+1} \neq \emptyset$ is called a **geodesic block-path**, from $b$ to $e$ beginning with $c_1$.*

**Definition 3.16.** *Let $\Upsilon_b$ be the set of irreducible blocks of minimal dimension beginning a geodesic block-path from $b$ to $e$.*

**Remark 3.17.** *In general, $\Upsilon_b$ is not of cardinal one. It is for $CAT(0)$ groups, but not for the Baumslag-Solitar group $BS(1,2) = \langle a^{\pm 1}, b^{\pm 1} \mid bab^{-1} = a^2 \rangle$.*

**Remark 3.18.** *Consider the group $\Gamma$ and its finite presentation $\langle S \mid R \rangle$, then we can complete the presentation as follows: let $T$ be a finite subset of $\Gamma$ with $T \cap S = \emptyset$, $T = T^{-1}$ and $e \notin T$, let $S' = T \cup S$ an amplified generating set and $R' = R \cup \{t = \bar{t} \mid t \in T\}$ where $\bar{t}$ is $t$ considered as a generator.
Then $\Gamma = \langle S' \mid R' \rangle$.*

**Lemma 3.19.** *We can choose $T$ such that if we build the blocks with the completed presentation $\langle S' \mid R' \rangle$, then every irreducible blocks are triangular, i.e. $\forall b \in \Gamma_n$, $card(b) = n + 1$. We call $\langle S' \mid R' \rangle$ a **triangularized presentation**.*

**Example 3.20.** *The complete triangularization: let $\Gamma = \langle S \mid R \rangle$ be a finitely presented group, then $\Gamma$ acts on $\Gamma_{max}$ (def. 3.9, 3.12); there are only finitely many orbits $O_1, ..., O_r$; choose $b_i \in O_i$; let $E_i = \{g \in \Gamma \mid g \in b_i\}$; let $T_i = \{gh^{-1} \mid g, h \in E_i, gh^{-1} \notin S \cup \{e\}\}$. Then amplifying the generating set with $T = \bigcup T_i$, we obtain obviously a triangularization called the complete triangularization. Note that this process increases the maximal dimension of the blocks. Note that card$(T)$ is finite because the group is finitely presented.*

## 4. Clifford algebra

We first quickly recall here the notion of Clifford algebra, for a more detailed exposition, see the course of A. Wassermann [7].

**Definition 4.1.** *For $V$ a $n$-dimensional Hilbert space, define the exterior algebra $\Lambda(V)$ equals to $\oplus_{k=0}^{n} \Lambda^k(V)$ with $\Lambda^0(V) = \mathbb{C}\Omega$. We called $\Omega$ the vacuum vector. Recall that $v_1 \wedge v_2 = -v_2 \wedge v_1$ so that $v \wedge v = 0$.*
*Note that $dim(\Lambda^k(V)) = C_n^k$ and $dim(\Lambda(V)) = 2^n$.*

**Definition 4.2.** *Let $\alpha_v$ be the creation operator on $\Lambda(V)$ defined by:*

$$\alpha_v(v_1 \wedge ... \wedge v_r) = v \wedge v_1 \wedge ... \wedge v_r \text{ and } \alpha_v(\Omega) = v$$

**Reminder 4.3.** *The dual $\alpha_v^\star$ is called the annihilation operator, then:*

$$\alpha_v^\star(v_1 \wedge ... \wedge v_r) = \sum_{i=0}^{r} (-1)^{i+1} (v, v_i) v_1 \wedge ... v_{i-1} \wedge v_{i+1} \wedge ... \wedge v_r \text{ and } \alpha_v^\star(\Omega) = 0$$

**Reminder 4.4.** *Let $\gamma_v = \alpha_v + \alpha_v^\star$, then $\gamma_v = \gamma_v^\star$ and $\gamma_v\gamma_w + \gamma_w\gamma_v = 2(v, w)I$.*

**Definition 4.5.** *The operators $\gamma_v$ generate the Clifford algebra Cliff$(V)$.*
*Note that the operators $\gamma_v$ are bounded and that Cliff$(V).\Omega = \Lambda(V)$.*

**Remark 4.6.** *$V$ admits the orthonormal basis $(v_a)_{a \in I}$.*
*We will write $\gamma_a$ instead of $\gamma_{v_a}$, so that $[\gamma_a, \gamma_{a'}]_+ = 2\delta_{a,a'}I$.*

Let $\Gamma$ be a finitely presented group, with a triangularized presentation $\langle S \mid R \rangle$.

**Definition 4.7.** *For any irreducible block $c$, let $\Delta_c = \{b \in \bigcup \Gamma_n \mid c \in \Upsilon_b\}$, with $\Upsilon_b$ defined on definition 3.16.*

**Remark 4.8.** *If $\Delta_c \neq \emptyset$ then $c \in \Delta_c$*
*$\bigcup \Gamma_n = \bigcup \Delta_c$ (it's not a partition in general)*
*If $\Delta_c \cap \Delta_{c'} \neq \emptyset$ with $c \neq c'$ then $dim(c) = dim(c')$ and $c.c'$ is connected.*

**Definition 4.9.** *Let $\bigsqcup_{\alpha \in \mathcal{J}} \mathcal{P}_\alpha$ be the **minimal partition** generated by $\bigcup \Delta_c$, ie $\bigsqcup_{\alpha \in \mathcal{J}} \mathcal{P}_\alpha = \bigcup \Gamma_n$, $\forall \alpha \in \mathcal{J}$, $\mathcal{P}_\alpha \neq \emptyset$ and $\exists n > 0$, $\exists c_1, ..., c_n$ irreducibles such that $\mathcal{P}_\alpha = \Delta_{c_1} \cap ... \cap \Delta_{c_1}$.*

**Lemma 4.10.** *For any $\alpha \in \mathcal{J}$, $\mathcal{P}_\alpha$ admits a unique element $c_\alpha$ (resp. $c'_\alpha$) of minimal dimension $m$ (resp. of maximal dimension $M$). Denote by $I_\alpha$ the set of blocks of dimension $m+1$ in $\mathcal{P}_\alpha$, then $\mathcal{P}_\alpha$ is in one-to-one correspondence with the power set $\mathcal{P}(I_\alpha)$; in particular, the cardinality of $\mathcal{P}_\alpha$ is $2^{M-m}$.*

**Definition 4.11.** *We naturally identify $\ell^2(\mathcal{P}_\alpha)$ with the exterior algebra $\Lambda(\ell^2(I_\alpha))$ on which operates the Clifford algebra Cliff$(\ell^2(I_\alpha))$ generated by $\gamma_a$, $a \in I_\alpha$.*

## 5. Dirac operator

**Definition 5.1.** *We define the **n-block lenght** $\ell_n(.)$ as follows: let $b$ be a block, then $\ell_n(b)$ is the minimal number of irreducible blocks decomposing a connected admissible $n$-dimensional block $c$ with $e \Subset c$ and, $b \Subset c$ or $b \cap c \neq \emptyset$.*

**Definition 5.2.** *Let $b$ be a block, then a sequence $(c_1, ..., c_{\ell_n(b)})$ with $b \Subset c_1$, $e \Subset c_{\ell(b)}$, $c_i \in \Gamma_n$ and $c_i \cap c_{i+1} \neq \emptyset$ is called a **geodesic n-block-path**, from $b$ to $e$ beginning with $c_1$.*

**Definition 5.3.** *For any $\alpha \in \mathcal{J}$, let $n = dim(c_\alpha) + 1$; for any $a \in I_\alpha$ define $p_a(\alpha)$ the number of geodesic $n$-block path from $c_\alpha$ to $e$ beginning with $a$; let $p(\alpha) = \sum_{a \in I_\alpha} p_a(\alpha)$; let $\lambda_a = \frac{p_a(\alpha)}{p(\alpha)} \ell_n(c_\alpha)$.*

**Definition 5.4.** *On $\ell^2(\mathcal{P}_\alpha) = \Lambda(\ell^2(I_\alpha))$, define the Dirac operator $D_\alpha$ by:*

$$D_\alpha = \sum_{a \in I_\alpha} \lambda_a . \gamma_a$$

**Remark 5.5.** *$\mathcal{P}_e := \Delta_e = \{e\}$, $\ell^2(\mathcal{P}_e) = \mathbb{C}e_1$, $I_e = \emptyset$ and $D_e = 0$.*

**Definition 5.6.** *Consider then the Hilbert space:*

$$\mathcal{H} = \bigoplus_n \ell^2(\Gamma_n) = \sum_c \ell^2(\Delta_c) = \bigoplus_{\alpha \in \mathcal{J}} \ell^2(\Delta_\alpha) = \bigoplus_{\alpha \in \mathcal{J}} \Lambda(\ell^2(I_\alpha))$$

$\mathbb{Z}_2$-graded by the decomposition into even and odd dimensional blocks:

$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$$

Define the unbounded selfadjoint operator $\mathcal{D} = \sum_{\alpha \in \mathcal{J}} D_\alpha$.

**Lemma 5.7.** $\mathcal{D}^2 = \sum_\alpha \lambda_\alpha^2 . p_\alpha$,
*with $\lambda_\alpha^2 = \sum_{a \in I_\alpha} \lambda_a^2$ and $p_\alpha$ the projection on $\ell^2(\Delta_\alpha)$.*

*Proof.* We use the orthonormal decomposition and the Clifford relations. □

**Proposition 5.8.** *$\mathcal{D}_+ : \mathcal{H}^+ \to \mathcal{H}^-$ is a Fredholm operator of index 1.*

**Proposition 5.9.** *$(\mathcal{D}^2 + I)^{-1}$ is compact.*
*For $t > 0$, the operator $e^{-t\mathcal{D}^2}$ is trass-class.*

**Definition 5.10.** *For any $g \in \Gamma$ and for any $s \in S$ define $p_s(g)$ the number of geodesic 1-block path from $g$ to $e$ beginning with $\{g, gs\}$; let $p(g) = \sum_{s\in S} p_s(g)$.*

**Definition 5.11.** *Let $\mathcal{C}$ be the class of finitely presented groups $\Gamma = \langle S \mid R\rangle$ such that $\forall g \in \Gamma$, $\exists K_g \in \mathbb{R}_+$ such that $\forall s \in S$ and $\forall h \in \Gamma$ (with $h, gh \neq e$):*

$$|\frac{p_s(gh)}{p(gh)} - \frac{p_s(h)}{p(h)}| \leq \frac{K_g}{\ell_1(h)}$$

**Examples 5.12.** *The class $\mathcal{C}$ is stable by direct or free product, it contains $\mathbb{Z}^n$, $\mathbb{F}_n$, the finite groups, and probably every amenable or automatic groups (containing the hyperbolic groups, see* [2]*).*

**Proposition 5.13.** *Let $\Gamma$ of class $\mathcal{C}$, $\mathcal{A} = C^\star_r(\Gamma)$ and $\mathcal{D}$ as previously then: $\{a \in \mathcal{A} \mid [\mathcal{D}, a] \in B(\mathcal{H})\}$ is dense in $\mathcal{A}$.*

**Theorem 5.14.** *$(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is an even $\theta$-summable spectral triple and $\mathcal{D}_+$ is index 1. It then gives a non-trivial element for the $K$-homology of $\mathcal{A}$.*

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